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# PARTITIONS WITH A RESTRICTION ON THE MULTIPLICITY OF THE SUMMANDS

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**Abstract.** Using the circle dissection method, a convergent series and several asymptotic formulae are obtained for $p(n, t)$, the number of partitions of the positive integer $n$ in which no part may be repeated more than $t$ times.

1. **Introduction.** If $n$ and $t$ are positive integers we shall denote by $p(n, t)$ the number of partitions of $n$ in which no summand appears more than $t$ times. In particular $p(n, 1)$ is the number of partitions of $n$ into unequal parts. Several authors have already studied $p(n, 1)$ (see [3], [6], [7]), and a convergent series and asymptotic formulae for this partition function are well known. In the present paper our objective is to generalize these results and obtain a convergent series representation and asymptotic formulae for $p(n, t)$ subject only to the restriction that $n \geqq t$. Our attack is based on the familiar circle dissection method of Hardy-Ramanujan-Rademacher.

2. **The transformation equation.** Since the time of Euler it has been known that the generating function of $p(n)$, the number of partitions of the positive integer $n$, is

$$F(x) = \prod_{m=1}^{\infty} (1-x^m)^{-1} = \sum_{n=0}^{\infty} p(n)x^n. \tag{2.1}$$

The reciprocal of $F(x)$ is

$$H(x) = \prod_{m=1}^{\infty} (1-x^m) = \sum_{n=0}^{\infty} P(n)x^n$$

where $P(n)$ represents the number of partitions of $n$ into an even number of distinct parts minus the number of partitions of $n$ into an odd number of distinct parts. We note (see Theorem 10.4 in [9]) that $P(n)=(-1)^j$ if $n=(3j^2 \pm j)/2$ for some $j=0, 1, 2, \ldots$ and $P(n)=0$ otherwise.

The generating function of $p(n, t)$ is easily seen to be

$$G(x, t) = \prod_{m=1}^{\infty} (1+x^m+x^{2m}+\cdots+x^{tm}) = \frac{F(x)}{F(x^{t+1})} = \sum_{n=0}^{\infty} p(n, t)x^n. \tag{2.2}$$

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For each of the three functions just mentioned we have convergence in the interior of the unit circle.

If $h$ and $k$ are relatively prime integers with $k>0$ and $z$ is a complex number with positive real part then it is known (see [5] or [10]) that

$$F(\exp\{2\pi ih/k-2\pi z/k\}) = z^{1/2}\omega(h,k)\exp\{\pi(1/z-z)/12k\}\cdot F(\exp\{2\pi ih'/k-2\pi/kz\}). \tag{2.3}$$

$hh'\equiv -1 \pmod{k}$ and $\omega(h,k)=\exp\{\pi is(h,k)\}$ where $s(h,k)$ is a Dedekind sum defined by $s(h,k)=\sum_{u=1}^{k}((u/k))((hu/k))$. $((v))=0$ if $v$ is an integer and $((v))=v-[v]-\frac{1}{2}$ otherwise.

With the aid of (2.3) we shall now derive a similar transformation equation for $G(x,t)$. In what follows $D=(k,t+1)$, $k=DK$, $t+1=DT$ where, of course, $(K,T)=1$. If we take

$$x = \exp\{2\pi ih/k-2\pi z/k\} \tag{2.4}$$

then $x^{t+1}=\exp\{2\pi iTh/K-2\pi Tz/K\}$.

Since $G(x,t)=F(x)/F(x^{t+1})$ it follows from (2.3) that

$$G(x,t) = T^{-1/2}w(h,k,t)\exp\{\pi(tz+(T-D)/Tz)/12k\}\cdot F(\exp\{2\pi ih'/k-2\pi/kz\})H(\exp\{2\pi ih^*/K-2\pi/TKz\}).$$

Here $Thh^*\equiv -1 \pmod{K}$, and

$$w(h,k,t) = \omega(h,k)/\omega(Th,K). \tag{2.5}$$

If $TT'\equiv 1 \pmod{K}$ where $T'$ is kept fixed and we let

$$y = \exp\{2\pi iT'h'/k-2\pi/Tzk\}, \tag{2.6}$$

then we verify without difficulty that $y^D=\exp\{2\pi ih^*/K-2\pi/TzK\}$, and $\exp\{2\pi ih'/k-2\pi/kz\}=y^T\exp\{2\pi ih'(1-TT')/k\}$. If $h'\equiv b \pmod{D}$ and $M\equiv d \pmod{D}$, where $1-TT'=MK$, then $\exp\{2\pi ih'(1-TT')/k\}=\exp\{2\pi ibd/D\}=e(b,d,D)$. Thus, we can write $\exp\{2\pi ih'/k-2\pi/kz\}=e(b,d,D)y^T$. If we define

$$J(y,t) = F(e(b,d,D)y^T)H(y^D) = \sum_{n=0}^{\infty}p(n)e^n(b,d,D)y^{Tn}\sum_{n=0}^{\infty}P(n)y^{Dn} = \sum_{n=0}^{\infty}c(n,b,d,D)y^n, \tag{2.7}$$

we have, finally, the following:

THEOREM 1. *If $x$ and $y$ are defined by* (2.4) *and* (2.6), *respectively,*

$$G(x,t) = T^{-1/2}w(h,k,t)\exp\{\pi(tz+(T-D)/Tz)/12k\}J(y,t). \tag{2.8}$$

3. **An exponential sum.** In what follows we shall require an estimate of the magnitude of a certain sum involving $w(h,k,t)$. We begin by stating a proposition concerning $\omega(h,k)$ whose proof appears in [4].

PROPOSITION 1. *If $k$ is odd then*

$$\omega(h, k) = (h|k)i^{(k-1)/2} \exp\{2\pi iq(h-h')/gk\}. \tag{3.1}$$

*If $k$ is even then*

$$\omega(h, k) = (k|h)i^{b(k+1)/2} \exp\{2\pi iq(h-h')/gk\}. \tag{3.2}$$

*$g=(3, k)$ or $g=8(3, k)$ according as $k$ is odd or even, $h'$ is any solution of $hh' \equiv -1 \pmod{gk}$, and $q$ is any solution of $fq \equiv 1 \pmod{gk}$ where $f=24/g$. In (3.2) $b \equiv h' \pmod 8$, and the branch of $i^{b(k+1)/2}$ is that corresponding to the principal value of the logarithm. $(a|c)$ is the Jacobi symbol.*

Our immediate objective is to obtain a result similar to this proposition for $w(h, k, t)$. We shall utilize some elementary properties of the Jacobi symbol (see Theorems 3.5, 3.6, 3.7 in [9]) and the fact that Proposition 1 obviously holds if $h, h', k, g, f, q, b$ are replaced by $Th, h^*, K, G, F, Q, B$, respectively. Three cases must be considered.

If $k$ is odd then, of course, $K$ is also odd. It follows from (2.5) and (3.1) that

$$w(h, k, t) = (h|k)(Th|K)i^{(k-K)/2} \exp\{2\pi iq(h-h')/gk\} \exp\{-2\pi iQ(Th-h^*)/GK\}.$$

If $T'$ is chosen so that $TT' \equiv 1 \pmod{GK}$ we easily verify that $h^* \equiv T'h' \pmod{GK}$. If $g=JG$ ($J=1$ or 3) then $F=Jf$ and $Q \equiv Aq \pmod{GK}$ where $JA \equiv 1 \pmod{GK}$. Also, $(h|k)(Th|K)=(h|D)(h|K)(h|K)(T|K)=(h|D)(T|K)$. We conclude that

$$w(h, k, t) = (h|D)(T|K)i^{(k-K)/2} \exp\{2\pi iq(Uh+Vh')/gk\} \tag{3.3}$$

where

$$U = 1-JA(t+1), \qquad V = JAT'D-1. \tag{3.4}$$

Note that $(h|D)(T|K)$ has absolute value one and depends *only* on $k$ and $t$ if we impose the restriction $h \equiv a \pmod D$ where $(a, D)=1$.

If $k$ and $K$ are both even then from (2.5) and (3.2) we have

$$\begin{aligned} w(h, k, t) = {} & (k|h)(K|Th)i^{b(k+1)/2}i^{-B(K+1)/2} \\ & \cdot \exp\{2\pi iq(h-h')/gk\} \exp\{-2\pi iQ(Th-h^*)/GK\}. \end{aligned}$$

Choosing $T'$ so that $TT' \equiv 1 \pmod{GK}$ we have $h^* \equiv T'h' \pmod{GK}$ and $B \equiv h^* \equiv T'b \pmod 8$ (since $8|G$). If $g=JG$ ($J=1$ or 3) then $F=Jf$ and $Q \equiv Aq \pmod{GK}$ where $A$ is defined as above. Also, if $D=2^\alpha D^*$ where $\alpha \geqq 0$ and $D^*$ is odd, then $(k|h)(K|Th)=(K|T)(2^\alpha|h)(D^*|h)=(K|T)(2^\alpha|h)(h|D^*)(-1)^{(h-1)(D^*-1)/4}$; where $(2^\alpha|h)=1$ if $\alpha$ is even and $(2^\alpha|h)=(-1)^{(h^2-1)/8}$ if $\alpha$ is odd. Therefore,

$$\begin{aligned} w(h, k, t) = {} & (K|T)(2^\alpha|h)(h|D^*)(-1)^{(h-1)(D^*-1)/4}i^{b(k+1)/2}i^{-T'b(K+1)/2} \\ & \cdot \exp\{2\pi iq(Uh+Vh')/gk\} \end{aligned} \tag{3.5}$$

where $U$ and $V$ are given by (3.4). We note that the coefficient of

$\exp\{2\pi iq(Uh+Vh')/gk\}$ has absolute value one and depends *only* on $k$ and $t$ if we impose the restrictions $h\equiv a \pmod D$ where $(a, D)=1$, and $h\equiv d \pmod 8$ where $d$ is odd.

If $k$ is even and $K$ is odd then

$$w(h, k, t) = (k|h)(Th|K)i^{b(k+1)/2}i^{-(K-1)/2} \cdot \exp\{2\pi iq(h-h')/gk\} \exp\{-2\pi iQ(Th-h^*)/GK\}.$$

As before $h^*\equiv T'h' \pmod{GK}$. If $g=JG$ ($J=8$ or 24) then $F=Jf$ and $Q\equiv Aq \pmod{GK}$ where $JA\equiv 1 \pmod{GK}$. Writing $D=2^\alpha D^*$ we have

$$\begin{aligned}(k|h)(Th|K) &= (D|h)(K|h)(h|K)(T|K)\\ &= (D|h)(T|K)(-1)^{(h-1)(K-1)/4}\\ &= (T|K)(2^\alpha|h)(h|D^*)(-1)^{(h-1)(D^*-1)/4}(-1)^{(h-1)(K-1)/4}\\ &= (T|K)(2^\alpha|h)(h|D^*)(-1)^{(h-1)(K-D^*)/4}.\end{aligned}$$

We conclude that

$$w(h, k, t) = (T|K)(2^\alpha|h)(h|D^*)(-1)^{(h-1)(K-D^*)/4}i^{b(k+1)/2}i^{-(K-1)/2} \cdot \exp\{2\pi iq(Uh+Vh')/gk\} \tag{3.6}$$

where $U$ and $V$ are given by (3.4). The coefficient of $\exp\{2\pi iq(Uh+Vh')/gk\}$ has absolute value one and depends only on $k$ and $t$ if $h\equiv a \pmod D$ where $(a, D)=1$, and $h\equiv d \pmod 8$ where $d$ is odd.

We summarize (3.3), (3.4), (3.5), (3.6) in the following proposition. All undefined symbols have the meanings given earlier in this section.

PROPOSITION 2. *$w(h, k, t)=C(h, k, t)\exp\{2\pi iq(Uh+Vh')/gk\}$ where $|C(h, k, t)|=1$. Furthermore, $C(h, k, t)$ depends only on $k$ and $t$ if $h\equiv a \pmod D$ where $(a, D)=1$ and also, if $k$ is even, $h\equiv d \pmod 8$ where $d$ is odd. If $g=JG$ ($J=1, 3, 8, 24$) and $JA\equiv 1 \pmod{GK}$, then $U=1-JA(t+1)$ and $V=JAT'D-1$.*

We are now prepared to prove the main result of this section.

THEOREM 2. *If $(k, t+1)=D$, $h\equiv a \pmod D$, $(a, D)=1$; $hh'\equiv -1 \pmod k$, $s_1\leqq h'<s_2 \pmod k$, $0\leqq s_1<s_2\leqq k$; $t\leqq n$; and $M$ is a fixed integer, then the sum*

$$Y = \sum_{h \bmod k}{}' w(h, k, t)\exp\{-2\pi i(hn-h'M)/k\}$$

*is subject to the estimate $O(n^{1/3}k^{2/3+\varepsilon})$ where the multiplicative constant implied by the O-symbol depends only on $t$. The symbol $\sum'$ indicates that the variable of summation runs through a reduced residue system of the given modulus subject, perhaps, to some other stated restrictions.*

**Proof.** Since $w(h, k, t)$ has period $k$ when viewed as a function of $h$, if we change

the modulus in $Y$ to $gk$ and select $h'$ so that $hh' \equiv -1 \pmod{gk}$, then by Proposition 2

$$Y = g^{-1} \sum_{h \bmod gk}' C(h, k, t) \exp\{2\pi i f(h)/gk\}$$

where $f(h) = (qU - gn)h + (qV + gM)h'$.

If $k$ is even we split $Y$ into four parts, $Y_1$, $Y_3$, $Y_5$, $Y_7$, so that in $Y_d$ we have $h \equiv d \pmod 8$ (as well as $h \equiv a \pmod D$). Then $Y = Y_1 + Y_3 + Y_5 + Y_7$ where

$$Y_d = C_d \sum_{h \bmod gk}' \exp\{2\pi i f(h)/gk\} \tag{3.7}$$

with $|C_d| = g^{-1} \leqq 1$.

If $k$ is odd then (3.7) holds if we identify $Y$ with $Y_d$ and ignore the restriction $h \equiv d \pmod 8$.

If we define the function $m(s)$ for all integers $s$ by requiring that $m(s) = 1$ if $s_1 \leqq s < s_2 \pmod k$, and $m(s) = 0$ otherwise, then $m(s)$ has period $k$. From the theory of finite Fourier series we have $m(s) = \sum_{j=0}^{k-1} \alpha_j \exp\{2\pi i s j/k\}$ where $\alpha_j = k^{-1} \sum_{s=0}^{k-1} m(s) \exp\{-2\pi i s j/k\}$. It is not difficult to prove (see §10 in [8]) that $\sum_{j=0}^{k-1} \alpha_j = O(\log k)$ so that $\sum_{j=0}^{k-1} \alpha_j = O(k^{\varepsilon})$ for any $\varepsilon > 0$.

We can now drop the restriction $s_1 \leqq h' < s_2 \pmod k$ and write

$$\begin{aligned} Y_d &= C_d \sum_{h \bmod gk}' m(h') \exp\{2\pi i f(h)/gk\} \\ &= C_d \sum_{j=0}^{k-1} \alpha_j \sum_{h \bmod gk}' \exp\{2\pi i((qU - gn)h + (qV + gM + gj)h')/gk\}. \end{aligned}$$

If $k$ is odd then $\sum'$ is a Kloosterman sum. If $k$ is even we write $D = 2^{\alpha} D^*$ where $\alpha \geqq 0$ and $D^*$ is odd. It is easy to see that if $\alpha \leqq 1$ then the two conditions (I) $h \equiv a \pmod D$ and (II) $h \equiv d \pmod 8$ are equivalent to a single condition of the form $h \equiv a^* \pmod{8D^*}$. If $\alpha = 2$ and $a \not\equiv d \pmod 4$ then $\sum'$ is empty. If $a \equiv d \pmod 4$ then (I) and (II) are equivalent to $h \equiv a^* \pmod{8D^*}$. If $\alpha \geqq 3$ and $a \not\equiv d \pmod 8$ then $\sum'$ is empty, while if $a \equiv d \pmod 8$ then (I) and (II) are equivalent to $h \equiv a \pmod D$. Thus, we see that in each case either $\sum'$ is empty or $\sum'$ is a Kloosterman sum. Using a theorem of Salié [12] it follows that

$$|Y| < C_0 k^{2/3+\varepsilon} (qU - gn, gk)^{1/3} \tag{3.8}$$

where $C_0$ is a constant which is independent of all the parameters involved.

Since $(f, gk) = 1$, we have $(qU - gn, gk) = (fqU - fgn, gk)$. But $fg = 24$ and $fqU = U + mgk$ where $m$ is an integer. Therefore, $(qU - gn, gk) = (U - 24n, gk) \leqq Dg(U - 24n, K)$. From Proposition 2 we see that $U = 1 - (t+1)(1 + pGK) = -t + PK$, so that $(qU - gn, gk) \leqq Dg(t + 24n, K) \leqq Dg(t + 24n) \leqq 25Dgn$. The conclusion of the theorem now follows from (3.8).

4. **A convergent series for $p(n, t)$.** Applying Cauchy's integral formula to $G(x, t)$ we have

$$2\pi i p(n, t) = \int_C x^{-n-1}G(x, t)\,dx = \sum_{h,k}{}' \int_{\xi_{hk}} x^{-n-1}G(x, t)\,dx.$$

Here $0 \leqq h < k \leqq N$, $(h, k)=1$, and $\xi_{hk}$ are the Farey arcs of order $N$ of $C$, the circle $|x| = \exp\{-2\pi N^{-2}\}$. If, on the arc $\xi_{hk}$, we let $x = \exp\{2\pi i h/k - 2\pi z/k\}$ where $z = wk$, $w = N^{-2} - i\theta$, we obtain

$$(4.1)\quad p(n, t) = \sum_{h,k}{}' \exp\{-2\pi i n h/k\} \int G(\exp\{2\pi i h/k - 2\pi z/k\}, t) \exp\{2\pi n w\}\,d\theta.$$

The limits of integration are $-1/k(k+k_1)$ and $1/k(k+k_2)$ where $k_1$, $k$, $k_2$ are the denominators of consecutive terms of the Farey series of order $N$.

If $D$ runs through the positive divisors of $t+1$, and $a$ runs through a reduced residue system modulo $D$, we have

$$(4.2)\qquad p(n, t) = \sum_{D \mid t+1} \sum_{a \bmod D}{}' \sum_{d=1}^{D} S(D, a, d)$$

where $S(D, a, d)$ denotes the sum of all those terms in (4.1) which satisfy the conditions $(k, t+1) = D$, $h \equiv a \pmod D$, and $M \equiv d \pmod D$ where $1 - TT' = MK$ (see the remarks just preceding (2.7)). Notice that if $ab \equiv -1 \pmod D$ and $hh' \equiv -1 \pmod k$, then $h' \equiv b \pmod D$ in $S(D, a, d)$. Now either $S(D, a, d) = 0$ or we have from (2.8), (2.7), (2.6)

$$\begin{aligned} S(D, a, d) = T^{-1/2} \sum_{h,k}{}' & w(h, k, t) \exp\{-2\pi i n h/k\} \\ & \cdot \int \sum_{j=0}^{\infty} c(j, b, d, D) \exp\{2\pi i h' T' j/k\} \\ & \cdot \exp\{-(\pi/k^2 wT)(2j - (T-D)/12) + \pi w(2n + t/12)\}\,d\theta. \end{aligned}$$

The limits of integration are as before, $1 \leqq k \leqq N$, $(k, t+1) = D$, $M \equiv d$, $h \equiv a$, $h' \equiv b$ (all modulo $D$) and $1 - TT' = MK$. We note that $c(j, b, d, D)$ depends only on $j$ here.

We now split $S(D, a, d)$ into two parts, $Q(D, a, d)$ and $R(D, a, d)$, according as $j < (T-D)/24$ or $j \geqq (T-D)/24$, respectively. Employing Rademacher's argument [11] and making use of Theorem 2 we find that

$$(4.3)\qquad R(D, a, d) = O(n^{1/3} N^{-1/3+\varepsilon} \exp\{2\pi n N^{-2}\}).$$

Here, and in the remainder of this section, the multiplicative constant implied by the $O$-notation depends at most on $t$.

In $Q(D, a, d)$ the condition that $j < (T-D)/24$ implies that $Q(D, a, d) = 0$ if $T \leqq D$. Since $DT = t+1$ this will occur if, and only if, $D \geqq (t+1)^{1/2}$. We therefore consider only those $D$ such that $D < (t+1)^{1/2}$.

Proceeding as in [11] we obtain

$$Q(D,a,d) = 2\pi T^{-1}\sum_{k}\sum_{j} c(j,b,d,D)A(k,t,n,j,a,T')L^*(k,t,n,j) + O(n^{1/3}N^{-1/3+\varepsilon}\exp\{2\pi nN^{-2}\}) \tag{4.4}$$

where $1\leqq k\leqq N$, $0\leqq j<(T-D)/24$, and the other restrictions mentioned earlier are still in force.

$$A(k,t,n,j,a,T') = \sum_{h \bmod k}{}' w(h,k,t)\exp\{-2\pi i(nh-T'jh')/k\}$$

where $h\equiv a \pmod D$.

$$L^*(k,t,n,j) = k^{-1}\{(T-D-24j)/(24n+t)\}^{1/2} \cdot I_1\{\pi(24n+t)^{1/2}(T-D-24j)^{1/2}/6kT^{1/2}\}$$

where $I_1(x)$ is the Bessel function of order one.

Since $D<T$ we see from (2.7) that if $j<(T-D)/24$ then $c(j,b,d,D)=P(j/D)$ if $D|j$ and $c(j,b,d,D)=0$ if $D\nmid j$. Therefore, if we let $j=Dm$ and write

$$J = (T-D)/24D, \qquad r = t/24, \tag{4.5}$$

we have from (4.2), (4.3), (4.4), first summing over $d$, $a$, and $D$, and then letting $N$ approach infinity,

THEOREM 3. *The number of partitions of the positive integer $n$ in which no part appears more than $t$ times has the following infinite series representation*:

$$p(n,t) = 2\pi(t+1)^{-1}\sum_{D}\sum_{k}\sum_{m<J} P(m)A(k,t,n,m)L(k,t,n,m). \tag{4.6}$$

*Here $D|(t+1)$ and $D<(t+1)^{1/2}$; $(k,t+1)=D$; $P(m)=0,\ 1,\ -1$, according to the rule given at the beginning of* §2;

$$A(k,t,n,m) = \sum_{h \bmod k}{}' w(h,k,t)\exp\{-2\pi i(nh-DT'mh')/k\}; \tag{4.7}$$

$$L(k,t,n,m) = D^{3/2}k^{-1}\{(J-m)/(n+r)\}^{1/2} \cdot I_1\{4\pi Dk^{-1}((J-m)(n+r)/(t+1))^{1/2}\} \tag{4.8}$$

*where $J$ and $r$ are given by* (4.5).

We remark that Theorem 3 can be given a different interpretation. For according to a theorem of Glaisher [1] $p(n,t)$ also represents the number of partitions of $n$ having the property that no part is divisible by $t+1$. From this point of view we also observe that Theorem 9 of [2] is the special case of Theorem 3 when $t+1$ is an odd prime.

5. **Some special cases.** If $t\leqq 24$ and $t=p^j-1$, where $p$ is a prime and $j=1$ or 2, then in (4.6) only $D=1$ and $m=0$ appear. Also, $T=p^j$ and $J=r=t/24$ so that we have

COROLLARY 3.1. *If* $t \leqq 24$ *and* $t = p^j - 1$, $p$ *a prime and* $j = 1$ *or* 2, *then*

$$p(n, t) = 2\pi p^{-j} \sum_k k^{-1}\{t/(t+24n)\}^{1/2} A(k, t, n, 0) I_1\{\pi(t^2+24nt)^{1/2}/(6kp^{j/2})\}$$

*where* $(p, k) = 1$.

If, in particular, $t = 1$, we have

COROLLARY 3.2. *The number of partitions of a positive integer* $n$ *into unequal parts is given by*

$$p(n, 1) = \pi \sum_k k^{-1}(24n+1)^{-1/2} A(k, 1, n, 0) I_1\{\pi(48n+2)^{1/2}/12k\}$$

*where* $2 \nmid k$.

This result agrees with Theorem 4 in [3]. (It is not difficult to show that $A(k, 1, n, 0)$ here and $B(k, n)$ in [3] are equal.)

6. **Asymptotic formulae.** In this section $c$ denotes a positive constant, and both $c$ and the multiplicative constant implied by the $O$-symbol depend at most on $t$.

If we write

(6.1) $r = t/24$,

(6.2) $G(m) = (r-m)^{1/2}$,

(6.3) $E = 4\pi(n+r)^{1/2}$,

(6.4) $s = (t+1)^{-1/2}$,

(6.5) $W = \sum_{m<r} P(m) G(m) I_1\{sEG(m)\}$,

then from (4.6) we have, splitting off the term for which $k = 1$,

$$p(n, t) = 2\pi s^2 W (n+r)^{-1/2}(1+S). \tag{6.6}$$

Here,

$$S = \sum_D \sum_{k>1} \sum_{m<J} P(m) A(k, t, n, m) D^{3/2}(kW)^{-1}(J-m)^{1/2} \cdot I_1\{DEsk^{-1}(J-m)^{1/2}\} \tag{6.7}$$

where $D \mid (t+1)$, $D < (t+1)^{1/2}$, $(k, t+1) = D$, and $J$ is given by (4.5).

We shall prove that for large $n$

$$S = O(\exp\{-cn^{1/2}\}) \tag{6.8}$$

which, in conjunction with (6.6), yields

THEOREM 4. *As* $n \to \infty$,

$$p(n, t) = 2\pi s^2 W (n+r)^{-1/2}(1 + O(\exp\{-cn^{1/2}\})) \tag{6.9}$$

*where* $r$, $s$, $W$ *are given by* (6.1), (6.4), (6.5), *respectively.*

For the proof of (6.8) we require two lemmas. The first is a restatement of some well-known results from the theory of Bessel functions. We shall prove the second.

LEMMA 1. *If $x$ is real and positive then*

(6.10) *$I_1(x)$ is a positive, monotonic increasing function of $x$,*

(6.11) *$I_1(x)=O(x)$ if $x<1$,*

(6.12) *$I_1(x)=e^x(2\pi x)^{-1/2}(1+O(x^{-1}))$ if $x>1$.*

LEMMA 2. *If $W$ is given by* (6.5) *then for large $n$*

$$W = r^{1/2}I_1\{sEr^{1/2}\}(1+O(\exp\{-cn^{1/2}\})). \tag{6.13}$$

**Proof.** We assume that $t>24$ since otherwise the result is immediate. From (6.5) we have

$$W = r^{1/2}I_1\{sEr^{1/2}\}\Big(1+\sum_{0<m<r} P(m)(1-m/r)^{1/2}I_1\{sEG(m)\}/I_1\{sEr^{1/2}\}\Big). \tag{6.14}$$

From (6.10), (6.12), (6.2), (6.3) it follows that for large $n$

$$\begin{aligned} I_1\{sEG(m)\}/I_1\{sEr^{1/2}\} &= O(\exp\{sE(G(1)-r^{1/2})\}) \\ &= O(\exp\{-4\pi sn^{1/2}(1+r/n)^{1/2}(r^{1/2}-(r-1)^{1/2})\}) \\ &= O(\exp\{-cn^{1/2}\}). \end{aligned} \tag{6.15}$$

Since $|P(m)(1-m/r)^{1/2}|<1$ (6.13) follows from (6.14) and (6.15), and the proof of the lemma is complete.

From (4.7) and Theorem 2 we see that $A(k, t, n, m)=O(n^{1/3}k^{2/3+\varepsilon})$. Therefore, from (6.7), (6.10), (6.13) and the fact that $|P(m)|\leqq 1$ we have for large $n$

$$S = O\Big(\sum_{k=2}^{\infty} n^{1/3}k^{-1/3+\varepsilon}I_1\{DEsJ^{1/2}k^{-1}\}/I_1\{sEr^{1/2}\}\Big).$$

But $DsJ^{1/2}=D(t+1)^{-1/2}((T-D)/24D)^{1/2}=T^{-1/2}((T-D)/24)^{1/2}<(24)^{-1/2}=\beta$, so that

$$S = O\Big(\sum_{k=2}^{\infty} n^{1/3}k^{-1/3+\varepsilon}I_1\{E\beta/k\}/I_1\{sEr^{1/2}\}\Big).$$

Splitting the sum over $k$ into two parts according as $k\leqq[E\beta]=X$ or $k>X$ and using Lemma 1 we have

$$\begin{aligned} S = {} & O\Big(\sum_{k=2}^{X} n^{1/3}k^{1/6+\varepsilon}\exp\{-E(sr^{1/2}-\beta/k)\}\Big) \\ & + O\Big(\sum_{k>X} n^{13/12}k^{-4/3+\varepsilon}\exp\{-Esr^{1/2}\}\Big). \end{aligned}$$

Since $sr^{1/2}=\beta(t/(t+1))^{1/2}\geqq\beta/2^{1/2}$ and $-\beta/k\geqq-\beta/2$ we see that $-E(sr^{1/2}-\beta/k)<-cn^{1/2}$ for large $n$. Also, $-Esr^{1/2}<-cn^{1/2}$. We now obtain easily

$$S = O(n^{11/12+\varepsilon}\exp\{-cn^{1/2}\}),$$

and (6.8) follows from the observation that $n=O(\exp\{.5cn^{1/2}\})$.

From Theorem 4 and Lemma 2 we have

COROLLARY 4.1. *As* $n \to \infty$

$$p(n, t) = 2\pi s^2 r^{1/2}(n+r)^{-1/2} I_1\{sEr^{1/2}\}(1+O(\exp\{-cn^{1/2}\})).$$

Finally, from Corollary 4.1 and (6.12) we obtain

COROLLARY 4.2. *As* $n \to \infty$

$$p(n, t) = 12^{1/2} s^{3/2} t^{1/4}(24n+t)^{-3/4} \exp\{sEr^{1/2}\}(1+O(n^{-1/2})).$$

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